



Stationary States for Fermions in an External Electromagnetic Field

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Outline

1.Introduction

2.Solutions of the Dirac equation in an electromagnetic field.

Solution in parallel electric and magnetic fields

Solution in perpendicular electric and magnetic fields

Solution in an arbitrary uniform electromagnetic field configuration

3.Feynman propagator in the electromagnetic field

4.Summary

Electromagnetic in heavy-ion collision

Effects on matter:

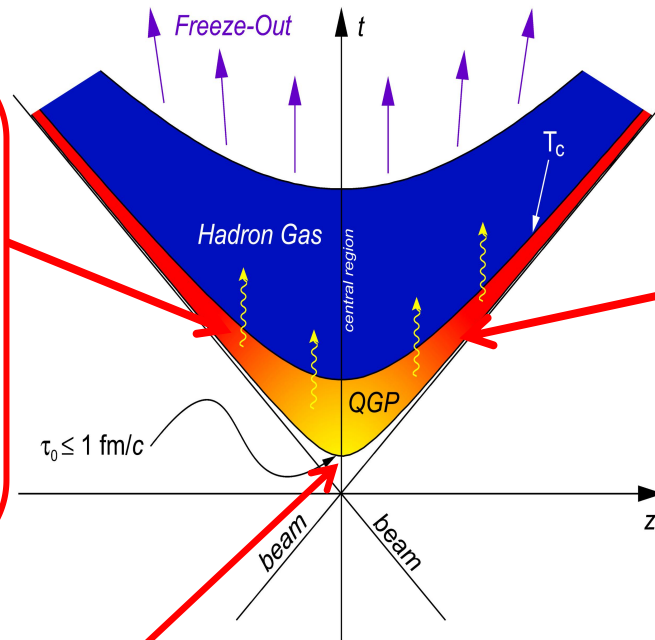
QCD phase structure:

黃梅

vorticity :

黃安平

.....



Effects on particles:

Polarization:

梁作堂, 王新年

.....

Particle production in the initial stage.

Strong electric and magnetic fields !

magnetic field:

[Chen S, Zhao J, Zhuang P. Journal of High Energy Physics, 2024, 2024\(9\): 1-21.](#)

electric field:

[Zhao X, Wang Y, Zhuang P. The European Physical Journal C, 2026, 86\(4\): 438](#)

How to consider the electric + magnetic field?

Basic idea

Solutions of the Dirac equation

$$(i\gamma^\mu D_\mu - m) \psi(t, \mathbf{x}) = 0.$$

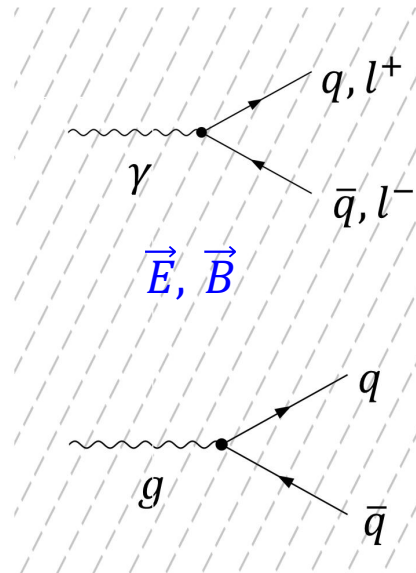


Feynman propagator, external legs

$$G(x - x')$$



feynman diagrams of SFQED, QCD in
strong eletromagnetic field



Solutions of the Dirac equation in parallel electric and magnetic field

Dirac equation under electromagnetic field:

$$(i\gamma^\mu D_\mu - m) \psi(t, \mathbf{x}) = 0 \quad D_\mu = \partial_\mu + ieA_\mu(t, \mathbf{x})$$

In static gauge:

$$A^\mu(t, \mathbf{x}) = (Ez, 0, -Bx, 0) \quad \psi(t, \mathbf{x}) = e^{-i(p_0 t - p_y y)} \phi(x, z)$$

$$[(p_0 - eEz) \gamma^0 + i\gamma^1 \partial_x - (p_y + eBx) \gamma^2 + i\gamma^3 \partial_z - m] \phi(x, z) = 0$$

Squaring the Dirac equation: Spinor equation $\longrightarrow$ Scalar equation

Key step in solving the equation!

$$\left[\partial_x^2 + \partial_z^2 + (p_0 - eEz)^2 - (p_y + eBx)^2 - m^2 + ieE\gamma^0\gamma^3 - ieB\gamma^1\gamma^2 \right] f(x, z)\chi = 0$$

the eight solutions of
the scalar equation:

$$\begin{aligned} f_{n,+}^+(\xi_1, \xi_2 | \pm) &= \nu_{n-1}(\xi_1) D_{i\mu_n/2} [\pm(1+i)\xi_2], \\ f_{n,-}^+(\xi_1, \xi_2 | \pm) &= \nu_{n-1}(\xi_1) D_{i\mu_n/2-1} [\pm(1+i)\xi_2], \\ f_{n,+}^-(\xi_1, \xi_2 | \pm) &= \nu_n(\xi_1) D_{i\mu_n/2} [\pm(1+i)\xi_2], \\ f_{n,-}^-(\xi_1, \xi_2 | \pm) &= \nu_n(\xi_1) D_{i\mu_n/2-1} [\pm(1+i)\xi_2]. \end{aligned}$$

$\nu_n(x)$: harmonic oscillator
wave function for $n = 0, 1, 2, \dots$

$D_v(z)$: parabolic cylinder
function

Solutions of the Dirac equation in parallel electric and magnetic field

Four linearly independent solutions:

$$\begin{aligned}
 \phi_{n,-}^+(x, z | \pm) &= [(p_0 - eEz) \gamma^0 + i\gamma^1 \partial_x - (p_y + eBx) \gamma^2 + i\gamma^3 \partial_z + m] f_{n,-}^+(\xi_1, \xi_2 | \pm) \chi_{\pm}^+ \\
 &= \pm(i-1)\sqrt{eE} \nu_{n-1}(\xi_1) D_{i\mu_n/2} [\pm(1+i)\xi_2] \chi_+^+ \\
 &\quad + i\sqrt{2neB} \nu_n(\xi_1) D_{i\mu_n/2-1} [\pm(1+i)\xi_2] \chi_-^- \\
 &\quad + m\nu_{n-1}(\xi_1) D_{i\mu_n/2-1} [\pm(1+i)\xi_2] \chi_-^+, \\
 \phi_{n,-}^-(x, z | \pm) &= [(p_0 - eEz) \gamma^0 + i\gamma^1 \partial_x - (p_y + eBx) \gamma^2 + i\gamma^3 \partial_z + m] f_{n,-}^-(\xi_1, \xi_2 | \pm) \chi_{\pm}^- \\
 &= \pm(i-1)\sqrt{eE} \nu_n(\xi_1) D_{i\mu_n/2} [\pm(1+i)\xi_2] \chi_+^- \\
 &\quad + i\sqrt{2neB} \nu_{n-1}(\xi_1) D_{i\mu_n/2-1} [\pm(1+i)\xi_2] \chi_-^+ \\
 &\quad + m\nu_n(\xi_1) D_{i\mu_n/2-1} [\pm(1+i)\xi_2] \chi_-^-.
 \end{aligned}$$

four independent bispinors:

$$\chi_+^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_+^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \chi_-^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \chi_-^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Solutions of the Dirac equation in perpendicular electric and magnetic field

In perpendicular electric and magnetic field, the method of squaring the Dirac equation is not working!

Lorentz transformation is working:

$$A^\mu = (Ez, 0, Bz, 0) \xrightarrow{\text{L.T.}} A'^\mu = (E'z', 0, 0, 0) \quad (E > B)$$

$$E' = \sqrt{E^2 - B^2}$$

Dirac equation:

$$(i\gamma^\mu D_\mu - m)\psi(x) = 0 \iff (i\gamma^\mu D'_\mu - m)\psi'(x') = 0$$

$$\psi(t, \mathbf{x}) = S^{-1}(\Lambda)\psi'(t', \mathbf{x}') \quad S^{-1}(\Lambda) = \exp\left(\frac{\eta}{2}\gamma^0\gamma^2\right)$$

$$\psi_{1,p}^\pm(t, \mathbf{x}) = \frac{e^{-i(p_0 t - p_x x - p_y y)}}{\sqrt{2}} \begin{pmatrix} \pm(i-1)\sqrt{eE'} \cosh \frac{\eta}{2} D_{i\lambda'/2}(\pm\zeta) + (m \cosh \frac{\eta}{2} - ip'_+ \sinh \frac{\eta}{2}) D_{i\lambda'/2-1}(\pm\zeta) \\ \pm(i-1)\sqrt{eE'} i \sinh \frac{\eta}{2} D_{i\lambda'/2}(\pm\zeta) + (-im \sinh \frac{\eta}{2} + p'_+ \cosh \frac{\eta}{2}) D_{i\lambda'/2-1}(\pm\zeta) \\ \pm(i-1)\sqrt{eE'} \cosh \frac{\eta}{2} D_{i\lambda'/2}(\pm\zeta) - (m \cosh \frac{\eta}{2} + ip'_+ \sinh \frac{\eta}{2}) D_{i\lambda'/2-1}(\pm\zeta) \\ \pm(i-1)\sqrt{eE'} i \sinh \frac{\eta}{2} D_{i\lambda'/2}(\pm\zeta) + (im \sinh \frac{\eta}{2} + p'_+ \cosh \frac{\eta}{2}) D_{i\lambda'/2-1}(\pm\zeta) \end{pmatrix}$$

$$\psi_{2,p}^\pm(t, \mathbf{x}) = \frac{e^{-i(p_0 t - p_x x - p_y y)}}{\sqrt{2}} \begin{pmatrix} \pm(i-1)\sqrt{eE'} i \sinh \frac{\eta}{2} D_{i\lambda'/2}(\pm\zeta) - (im \sinh \frac{\eta}{2} + p'_- \cosh \frac{\eta}{2}) D_{i\lambda'/2-1}(\pm\zeta) \\ \pm(i-1)\sqrt{eE'} \cosh \frac{\eta}{2} D_{i\lambda'/2}(\pm\zeta) + (m \cosh \frac{\eta}{2} + ip'_- \sinh \frac{\eta}{2}) D_{i\lambda'/2-1}(\pm\zeta) \\ \mp(i-1)\sqrt{eE'} i \sinh \frac{\eta}{2} D_{i\lambda'/2}(\pm\zeta) - (im \sinh \frac{\eta}{2} - p'_- \cosh \frac{\eta}{2}) D_{i\lambda'/2-1}(\pm\zeta) \\ \mp(i-1)\sqrt{eE'} \cosh \frac{\eta}{2} D_{i\lambda'/2}(\pm\zeta) + (m \cosh \frac{\eta}{2} - ip'_- \sinh \frac{\eta}{2}) D_{i\lambda'/2-1}(\pm\zeta) \end{pmatrix}$$

Solution in an arbitrary uniform electromagnetic field configuration

Lorentz transformation can transform any constant uniform electromagnetic field into electric and magnetic fields that are mutually parallel

uniform electromagnetic field

parallel electric and magnetic field

L.T.



$$A^\mu = (Ez, 0, B_x z - B_z x, 0)$$

$$A''^\mu = \left(\sqrt{E_x'^2 + E_z'^2} z'', 0, -\sqrt{B_x'^2 + B_z'^2} x'', 0 \right)$$

$$\psi_{n,p}(t, \mathbf{x}) = S^{-1}(\Lambda_B) S^{-1}(\Lambda_R) \psi''_{n,p''}(t'', \mathbf{x}'')$$

inverse boost operator:

$$S^{-1}(\Lambda_B) = \cosh\left(\frac{\tilde{\eta}}{2}\right) I + \sinh\left(\frac{\tilde{\eta}}{2}\right) \gamma^0 \gamma^2$$

inverse rotation operator:

$$S^{-1}(\Lambda_R) = \cos\left(\frac{\theta}{2}\right) I - \sin\left(\frac{\theta}{2}\right) \gamma^1 \gamma^3$$

$$\psi_{n,p}(t, \mathbf{x}) = S^{-1}(\Lambda_B) S^{-1}(\Lambda_R) \psi''_{n,p''}(t'', \mathbf{x}'')$$

Some discussions of bound states

Asymptotic behavior at infinity $\longrightarrow$ The presence or absence of bound states

At $z \rightarrow \infty$

$$D_{i\lambda/2}(+\xi_+) = \Delta_1(\xi) = e^{-\pi\lambda/8} (2\xi^2)^{i\lambda/4} e^{-i\xi^2/2}$$

$$D_{i\lambda/2}(-\xi_+) = e^{\pi\lambda/2} \Delta_1(\xi)$$

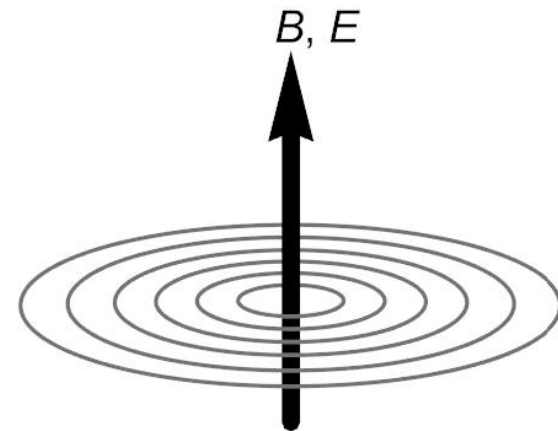
$$D_{i\lambda/2-1}(+\xi_+) = 0$$

$$D_{i\lambda/2-1}(-\xi_+) = \Delta_2(\xi) = \frac{\sqrt{2\pi}}{\Gamma(1-i\lambda/2)} e^{\pi\lambda/8} (2\xi^2)^{-i\lambda/4} e^{i\xi^2/2}$$

At $x \rightarrow \infty$

$$\nu_n(\xi_1) \rightarrow 0$$

For parallel electric and magnetic fields, magnetic field gives rise to a Landau-level-like structure in the transverse direction, whereas an electric field does not produce bound states in the longitudinal direction.



$$p_0^2 = 2neB + m^2 + p_{0,\parallel}^2$$

$$p_{0,\parallel}^2 = \hat{H}^2 - \left(-\partial_x^2 - (\partial_y + ieBx)^2 - eB\Sigma_z \right) - m^2$$

Feynman propagator in the electromagnetic field

Feynman propagator:

$$(i\gamma^\mu D_\mu - m) G(x, x') = \delta^{(4)}(x - x')$$

Feynman propagator of free particles:

$$G = -i \langle 0 | T \hat{\psi}(x) \hat{\bar{\psi}}(x') | 0 \rangle / \langle 0 | 0 \rangle$$

Important effects of electric fields: Schwinger pair production!

in-vacuum and out-vacuum are not equal:

$$|0, in\rangle \neq |0, out\rangle$$

This is why a global definition of particles and antiparticles is not possible in an electric field.

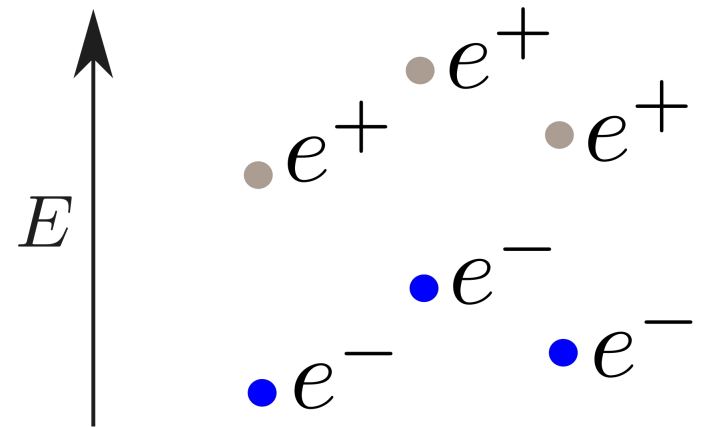
Feynman propagator in external field:

$$G = -i \langle 0, out | T \hat{\psi}(x) \hat{\bar{\psi}}(x') | 0, in \rangle / \langle 0, out | 0, in \rangle$$

Bogoliubov transformation relates the in-vacuum and the out-vacuum:

$$\begin{aligned} a_q^{out} &= c_1 a_q^{in} - c_2^* b_q^{\dagger in} \\ b_q^{\dagger out} &= c_2 a_q^{in} + c_1^* b_q^{\dagger in} \end{aligned}$$

$$|c_1|^2 + |c_2|^2 = 1$$



Feynman propagator in the electromagnetic field

The field operators can be defined on the in-states and out-states, respectively.

$$\begin{aligned}\hat{\psi}(x) &= \sum_{p_y, n, s, a, p_0} \left(a_q^{\text{in}} u_q^{\text{in}} e^{-ip_0 t + ip_y y} + b_q^{\dagger \text{in}} v_q^{\text{in}} e^{ip_0 t - ip_y y} \right) \\ &= \sum_{p_y, n, s, a, p_0} \left(a_q^{\text{out}} u_q^{\text{out}} e^{-ip_0 t + ip_y y} + b_q^{\dagger \text{out}} v_q^{\text{out}} e^{ip_0 t - ip_y y} \right)\end{aligned}$$

When considering parallel electric and magnetic fields:

How to determine the relationship between the in-vacuum and the out-vacuum?



Asymptotic behavior of solutions of the Dirac equation at infinity!

$$\phi_{n,-}^s(x, z \rightarrow +\infty | +) \propto \Delta_1$$

$$\phi_{n,-}^s(x, z \rightarrow -\infty | -) \propto \Delta_1$$

$$\phi_{n,-}^s(x, z \rightarrow +\infty | -) + e^{\pi\mu_n/2} \phi_{n,-}^s(x, z \rightarrow +\infty | +) \propto \Delta_2$$

$$\phi_{n,-}^s(x, z \rightarrow -\infty | +) + e^{\pi\mu_n/2} \phi_{n,-}^s(x, z \rightarrow -\infty | -) \propto \Delta_2$$

$$\Delta_1(\xi) = e^{-\pi\lambda/8} (2\xi^2)^{i\lambda/4} e^{-i\xi^2/2}$$

$$\Delta_2(\xi) = \frac{\sqrt{2\pi}}{\Gamma(1-i\lambda/2)} e^{\pi\lambda/8} (2\xi^2)^{-i\lambda/4} e^{i\xi^2/2}$$

Feynman propagator in the electromagnetic field

The particles and antiparticles in the in- and out-states are obtained from their asymptotic forms at infinity:

$$\begin{aligned}
 \text{in-positive} \quad & -\psi_{n,p_0,p_y}^s(t, \mathbf{x}) = e^{-i(p_0 t - p_y y)} \phi_{n,-}^s(x, z | +), \\
 \text{out-negative} \quad & +\psi_{n,p_0,p_y}^s(t, \mathbf{x}) = e^{-i(p_0 t - p_y y)} \phi_{n,-}^s(x, z | -), \\
 \text{out-positive} \quad & +\psi_{n,p_0,p_y}^s(t, \mathbf{x}) = e^{-i(p_0 t - p_y y)} \left[\phi_{n,-}^s(x, z | -) + e^{\frac{\pi}{2} \mu_n} \phi_{n,-}^s(x, z | +) \right], \\
 \text{in-negative} \quad & -\psi_{n,p_0,p_y}^s(t, \mathbf{x}) = e^{-i(p_0 t - p_y y)} \left[\phi_{n,-}^s(x, z | +) + e^{\frac{\pi}{2} \mu_n} \phi_{n,-}^s(x, z | -) \right],
 \end{aligned}$$

$p_0 \sim (-\infty, +\infty)$

$$\begin{aligned}
 u_q^{\text{in}} &= \frac{1}{N_1} \phi_{n,-}^s(x, z | +), \\
 u_q^{\text{out}} &= \frac{1}{N_2} \left(\phi_{n,-}^s(x, z | -) + e^{\pi \mu_n / 2} \phi_{n,-}^s(x, z | +) \right), \\
 v_q^{\text{in}} &= \frac{1}{N_2} \left(\phi_{n,-}^s(x, z | +) + e^{\pi \mu_n / 2} \phi_{n,-}^s(x, z | -) \right) |_{p_0 \rightarrow -p_0, p_y \rightarrow -p_y}, \\
 v_q^{\text{out}} &= \frac{1}{N_1} \phi_{n,-}^s(x, z | -) |_{p_0 \rightarrow -p_0, p_y \rightarrow -p_y}.
 \end{aligned}$$

$$N_1 = \frac{\sqrt{(2\pi)^2 2eE}}{e^{-3\pi \mu_n / 8}}$$

$$|N_1/N_2|^2 = 1/(e^{\pi \mu_n} - 1)$$

$$(v_{q_1}^{\text{out}}, u_{q_2}^{\text{in}}) = -e^{-\pi \mu_n / 2} \delta_{q_1 q_2}$$

Bogoliubov transformation coefficient

$$|c_1|^2 = 1 - e^{-\pi \mu_n}$$

$$c_2 = e^{-\pi \mu_n / 2}$$

Feynman propagator in the electromagnetic field

Use Gaussian wave packets to verify the **orthonormality and completeness**:

The key difficulties: the stationary solutions are **not square-integrable**, and the integrals involving parabolic cylinder functions are difficult to evaluate.

Gaussian wave packet: let the wave function is

square-integrable

$$\Phi_{\sigma, \bar{p}_0}^{(r)}(t, z) = \int_{-\infty}^{+\infty} dp_0 G_{\sigma, \bar{p}_0}(p_0) e^{-ip_0 t} F_{p_0}^{(r)}(z)$$

$$\Phi_{\sigma, \bar{p}_0}^{(r)}(t, \cdot) \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

$$G_{\sigma, \bar{p}_0}(p_0) \xrightarrow[\sigma \rightarrow 0^+]{S'} \delta(p_0 - \bar{p}_0)$$

$$\begin{aligned} & \geq_{n, s; \sigma, \rho; \bar{p}_0, \bar{p}_y}^{(r)}(t, \mathbf{x}) \\ &= \int_{-\infty}^{+\infty} dp_0 \int_{-\infty}^{+\infty} dp_y G_{\sigma, \bar{p}_0}(p_0) G_{\rho, \bar{p}_y}(p_y) \Psi_{n, s, p_0, p_y}^{(r)}(t, \mathbf{x}) \end{aligned}$$

$$\sigma, \rho \rightarrow 0^+$$



$$\begin{aligned} & \sum_{(n, s) \in \mathcal{I}} \int_{-\infty}^{+\infty} dp_y \int_{-\infty}^{+\infty} dp_0 \left[\Psi_{n, p_0, p_y}^{-s}(t, \mathbf{x}) \Psi_{n, p_0, p_y}^{-s\dagger}(t, \mathbf{x}') \right. \\ & \quad \left. + \Psi_{-n, p_0, p_y}^s(t, \mathbf{x}) \Psi_{-n, p_0, p_y}^{s\dagger}(t, \mathbf{x}') \right] \\ &= \delta^{(3)}(\mathbf{x} - \mathbf{x}') I_4. \end{aligned}$$

Feynman propagator in the electromagnetic field

Feynman propagator:

$$G(x, x') = \frac{-i}{c_1^*} \sum_q \left[\theta(t - t') u_q^{\text{out}}(x, z|a) \bar{u}_q^{\text{in}}(x', z'|a) e^{-ip_0(t-t') + ip_y(y-y')} \right. \\ \left. - \theta(t' - t) v_q^{\text{in}}(x, z|a) \bar{v}_q^{\text{out}}(x', z'|a) e^{-ip_0(t'-t) + ip_y(y'-y)} \right]$$

The Feynman propagator we have obtained is well-defined and also corresponds well to the Schwinger proper-time method.

$$LG(p_0, \mathbf{x}, \mathbf{x}') = \frac{1}{c_1^*} \sum_{q, \epsilon} \left[u_{q, \epsilon}^{\text{out}}(x, z|a) \bar{u}_{q, \epsilon}^{\text{in}}(x', z'|a) e^{ip_y(y-y')} \right. \\ \left. + v_{q, \epsilon}^{\text{in}}(x, z|a) \bar{v}_{q, \epsilon}^{\text{out}}(x', z'|a) e^{ip_y(y'-y)} \right].$$



$$LG(p_0, \mathbf{x}, \mathbf{x}') = \gamma^0 \delta(\mathbf{x} - \mathbf{x}')$$

$$L = p_0 - H$$

Summary

1. Analytical Dirac solutions are obtained for arbitrary constant uniform electromagnetic fields.

2. A well-defined fermion propagator is constructed, consistent with Schwinger's proper-time result.

3. Next: apply this framework to scattering processes in early-stage heavy-ion collisions.

Thank you for your attention!